Elliptic delsarte surfaces

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Chapter 3

A classification of Delsarte surfaces.

The goal of this chapter is to create a classification of all elliptic Delsarte surfaces. We will do this by attaching to each elliptic curve, corresponding to a Delsarte surface, a polygon, its Newton polygon. Up to equivalence there are only finitely many polygons that come from a genus 1 curve. For each such polygon we classify the curves corresponding to a Delsarte surface, with that polygon attached to it. An abbreviated version of this chapter and the next chapter appeared in [8].

3.1 Newton Polygons.

We start with some definitions.

Definition 3.1.1. An *integral polygon* or just a *polygon* is the convex hull in $\mathbb{R}^2$ of a finite subset of $\mathbb{Z}^2$.

Definition 3.1.2. Let $f = \sum_{(a,b)} \alpha_{(a,b)} X^a Y^b$ be a Laurent polynomial. Let S be the finite set defined by $S = \left\{(a,b) \in \mathbb{Z}^2 : \alpha_{(a,b)} \neq 0\right\}$. Define the *Newton polygon*, $\Gamma(f)$, of f as the convex hull of S.

Note that for our purposes we only need $f \in k[X,Y]$. We have defined the Newton polygon over the Laurent polynomials because the definition is more natural this way.

Our idea is to let f define a curve and then use the Newton polygon to say something about the genus of such a curve. For this to work we will require that f is nondegenerate with respect to its Newton polygon

Definition 3.1.3. Take $f = \sum_{(a,b)\in S} \alpha_{(a,b)} X^a Y^b \in k[X^{\pm 1}, Y^{\pm 1}]$. For every edge γ of the Newton polygon define $f_\gamma = \sum_{(a,b)\in S\cap\gamma} \alpha_{(a,b)} X^a Y^b$. We have that f is nondegenerate with respect to its Newton polygon if for every γ we have f_γ, $\frac{\partial f_\gamma}{\partial X}$ and $\frac{\partial f_\gamma}{\partial Y}$ generate the unit ideal in $k[X^{\pm 1}, Y^{\pm 1}]$.

Note that f is nondegenerate with respect to its Newton polygon if all f_γ only have simple roots as polynomials in X and Y. The following theorem given in [1] shows how we can use the Newton polygon to deduce the genus of an elliptic curve.

Theorem 3.1.4. *Let $f(X,Y) \in k[X^{\pm 1}, Y^{\pm 1}]$ be absolutely irreducible. Let g be the geometric genus of C. Then*

$$g \leq \#\{\textit{integral points in the interior of } \Gamma(f)\}.$$

Equality holds if f is nondegenerate with respect to its Newton polygon and the singular points of the projective closure of C in $\mathbb{P}^2$ are all among $(0:0:1)$, $(0:1:0)$ and $(1:0:0)$.

Proof. See [1, Theorem 4.2] □

The (in)equality given here is called *Baker's formula* and was originally proven by H.F Baker in 1893. Our plan is to use this formula to find all genus one curves given by an equation of the form f by looking at all polygons with exactly one interior point.

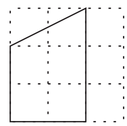


Figure 3.1: The Newton polygon of $1 + 3x^2 + y^2 + x^2y^3$.

Example 3.1.5. Consider the projective curve corresponding to $f = 1 + 3x^2 + y^2 + x^2y^3$. We first find that the only singular points are the points $(1:0:0)$ and $(0:1:0)$. See figure 3.1 for the Newton polygon. We have to check that the curve is non-singular with respect to its Newton polygon. To do this we consider $f_{\gamma_1} = 1 + 3x^2$ and see that $\frac{\partial f_{\gamma_1}}{\partial x} = 6x$ already generates the unit ideal. Likewise for $f_{\gamma_2} = 1 + y^2$, $\frac{\partial f_{\gamma_2}}{\partial y} = 2y$ generates the unit ideal. For $f_{\gamma_3} = y^2 + x^2y^3$, $\frac{\partial f_{\gamma_3}}{\partial x} = 2xy^3$ generates the unit ideal, and finally for $f_{\gamma_4} = 3x^2 + x^2y^3$ the unit ideal is generated by $\frac{\partial f_{\gamma_4}}{\partial y} = 3x^2y^2$. We conclude that the curve satisfies all criteria for equality in Baker's formula. Since its Newton polygon has two interior points, the genus of this curve is 2.

3.2 Curves corresponding to Delsarte surfaces and Baker's formula

In the previous section we gave Baker's formula. We are interested in applying Baker's formula for curves defined by the sum of four monomials. In this case the inequality is in fact an equality, as is stated in the following theorem:

Theorem 3.2.1. *Let C be a curve over the field $k(t)$, defined by an absolutely irreducible polynomial of the form given in (2.1). Also assume that* $\det(A) \neq 0$, *then the genus of C equals the number of interior lattice points of its Newton polygon.*

Proof. The theorem follows simply from theorem 3.1.4 and the following two lemma's. □

Lemma 3.2.2. *Let C be the projective closure in $\mathbb{P}^2$ of a curve over $k(t)$ defined by a polynomial f as in (2.1). Assume that A is nonsingular. Then C does not* *have singular points outside the point* $(0:0:1)$, $(0:1:0)$ *and* $(1:0:0)$ *over* $\overline{k(t)}$.

Proof. The curve C is given projectively by the polynomial:

$$\tilde{f} = \sum_{i=0}^{3} t^{a_{i0}} X^{a_{i1}} Y^{a_{i2}} Z^{m-a_{i1}-a_{i2}},$$

where m is a positive integer. Let B be as defined in theorem 2.1.3, so

$$B = \begin{pmatrix} a_{00} & a_{01} & a_{02} & m-a_{01}-a_{02} \\ a_{10} & a_{11} & a_{12} & m-a_{11}-a_{12} \\ a_{20} & a_{21} & a_{22} & m-a_{21}-a_{22} \\ a_{30} & a_{31} & a_{32} & m-a_{31}-a_{32} \end{pmatrix}.$$

Since A is nonsingular we know that B is likewise nonsingular. We define $B_{1,1}$ to be the matrix obtained by deleting the first row and column from B. Since B is nonsingular, we can without loss of generality assume that $\det(B_{1,1}) \neq 0$ and $a_{00} \neq 0$.

We claim that there exist a, b_1, b_2, $b_3 \in \mathbb{Z}$ with the following property. With $s \in \overline{k(t)}$ satisfying $s^a = t$, the map $(X:Y:Z) \mapsto (s^{b_1}X : s^{b_2}Y : s^{b_3}Z)$ defined over $k(s) \supset k(t)$ defines an isomorphism from C to $\tilde{C}$. Here $\tilde{C}$ is the curve given by:

$$\tilde{f} := s^n X^{a_{01}} Y^{a_{02}} Z^{m-a_{01}-a_{02}} + \sum_{i=1}^{3} X^{a_{i1}} Y^{a_{i2}} Z^{m-a_{i1}-a_{i2}},$$

for some $n \in \mathbb{Z}_{>0}$.

The proof of this claim is an exercise in linear algebra, as follows: we look for α, β, $\gamma \in \mathbb{Q}$ such that

$$(t^\alpha X)^{a_{i1}} (t^\beta Y)^{a_{i2}} (t^\gamma Z)^{m-a_{i1}-a_{i2}} = t^{a_{i0}} X^{a_{i1}} Y^{a_{i2}} Z^{m-a_{i1}-a_{i2}}$$

for $i = 1, 2, 3$. This means that $v = (\alpha, \beta, \gamma)^T$ is a solution of

$$B_{1,1} v = \begin{pmatrix} a_{10} \\ a_{20} \\ a_{30} \end{pmatrix}.$$

Since by assumption $\det(B_{1,1}) \neq 0$, the solution v exists and is unique.

If v is the zero vector then we take $a = 1$ and $b_1 = b_2 = b_3 = 0$. In all other cases we pick $a \in \mathbb{Z}$ the greatest common divisor of the denominators of α, β and γ. We then pick $b_1 = \alpha a$, $b_2 = \beta a$ and $b_3 = \gamma a$.

Let B_0 the the first column of B and let B' be the matrix obtained by deleting the first column of B. Then since B is nonsingular we find $B'v \neq B_0$. This means precisely that $n \neq 0$. By possibly flipping the sign of a we can ensure that $n > 0$.

To prove the lemma, it suffices to prove that $\tilde{C}$ has no singular points outside $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$. Assume $(x:y:z)$ is a singular point on $\tilde{C}$. We construct the row vector $w = (w_0, w_1, w_2, w_3)$. Here $w_0 = s^n x^{a_{01}} y^{a_{02}} z^{m-a_{01}-a_{02}}$ and for $i \in \{1,2,3\}$ we put $w_i = x^{a_{i1}} y^{a_{i2}} z^{m-a_{i1}-a_{i2}}$.

Note that $wB' = 0$. There are two possibilities. The first possibility is that $w_0 = 0$. Since $\det(B_{1,1}) \neq 0$ this implies $w = (0,0,0,0)$. It follows that $(x:y:z)$ is one of $(1:0:0)$, $(0:1:0)$ or $(0:0:1)$.

If $w_0 \neq 0$ we can assume $w_0 = 1$. Since $\det(B_{1,1}) \neq 0$ we find that w is uniquely determined and $w \in \mathbb{Q}^4$. We have to again subdivide in a few cases.

If all the w_i are nonzero then the coordinates x, y and z are also nonzero. Take $e = (1,0,0)^T$ and let $u = (u_1, u_2, u_3)^T$ be the solution to $B_{1,1}u = e$. Then we find:

$$\prod_{i=1}^{3} w_i^{u_i} = \prod_{i=1}^{3} (x^{a_{i1}} y^{a_{i2}} z^{m-a_{i1}-a_{i2}})^{u_i} = x.$$

Since we have $w_i \in \mathbb{Q}$ we find that x is algebraic over $\mathbb{Q}$, so $x \in k$. The same applies for y and z. This however contradicts the fact that $w_0 = 1$. We conclude that this case cannot occur.

If exactly one of the w_i is zero, this implies that exactly one of the coordinates is zero. In that case we can do exactly the same as in the previous case, but with one dimension less.

The final case is that exactly two of the w_i are zero, but with only one coordinate zero. Without loss of generality we can assume that $w_1 = w_2 = 0$ and $x = 0$. The polynomial $\tilde{F}$ now has the form:

$$\tilde{f} = s^n Y^{a_{02}} Z^{m-a_{02}} + X^{a_{11}} Y^{a_{12}} Z^{m-a_{11}-a_{12}} + X^{a_{21}} Y^{a_{22}} Z^{m-a_{21}-a_{22}} + Y^{a_{32}} Z^{m-a_{32}},$$

with $a_{11}, a_{21} > 0$. We are looking for a singular point of the curve defined by $\tilde{f}$ with $X = 0$. Looking at $\tilde{f}$ and $\frac{\partial \tilde{f}}{\partial Y}$ we quickly see that this implies $a_{02} = a_{32}$. We then conclude that the only possible singular points are given by either $Y = 0$ or $Z = 0$. □

Lemma 3.2.3. *Let f be as in (2.1). Assume that $\det(A) \neq 0$, then f is nondegenerate with respect to its Newton polygon.*

Proof. For any edge γ we find f_γ has either two or three terms. Four terms on one edge is not possible, since then $\det(A) = 0$. The case where f_γ has only two terms is simple, since $\mathrm{char}(k) = 0$. (In fact a simpler version of the argument below suffices.)

We will only do the case where f_γ has three terms. Without loss of generality we can assume that $f_\gamma = X^a + Y^b + s^n X^{\lambda a} Y^{(1-\lambda)b}$, where s a root of t. Here n is nonzero, since otherwise $\det(A) = 0$. Define $\eta = X^a / Y^b$. Now we assume that $a \neq 0$, then $\frac{\partial f_\gamma}{\partial X} = 0$ and $f_\gamma - \frac{X}{a}\frac{\partial f_\gamma}{\partial X} = 0$ gives the system of equations

$$\begin{cases} \eta + \lambda s^n \eta^\lambda = 0, \\ 1 + (1-\lambda) s^n \eta^\lambda = 0. \end{cases}$$

Since s is transcendental over k this has no solution. □

3.3 The forms of the Newton polygon

Definition 3.3.1. We call two polygons A, B *integrally equivalent* if B is the image of A under a linear map given by a matrix in $\mathrm{GL}_2(\mathbb{Z})$ possibly composed with a translation over an element of $\mathbb{Z}^2$.

Remark 3.3.2. Being integrally equivalent is an equivalence relation.

We recall a result from [13].

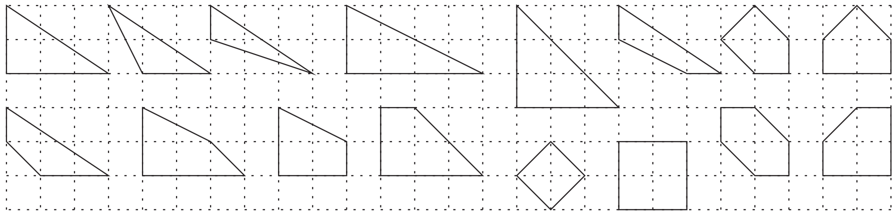


Figure 3.2: All polygons with exactly one interior point up to equivalence.

Theorem 3.3.3. *Up to integral equivalence there are only 16 polygons with exactly one interior point. These polygons are given in figure 3.2.*

Proof. This is the main result of [13]. It should be said that the original article contained a mistake, that has been corrected in 2005 (see p.12 of [12]). □

If $f \in k[X,Y]$ is irreducible and not a multiple of X or Y, then $\Gamma(f)$ has at least one point on both the x and y-axis. Furthermore $\Gamma(f)$ is contained in the first quadrant. Any integral polygon can be shifted in a unique way such that it satisfies these criteria (being contained in the first quadrant and with a point on both axes). We shall consider this to be the *default position* of the polygon.

Proposition 3.3.4. *Let $f(X,Y) = \sum_{(a,b)\in S} \alpha_{(a,b)} X^a Y^b$ be an irreducible polynomial defining the curve C. Assume that all $\alpha_{(a,b)} \neq 0$. Given a polygon A, in default position, integrally equivalent to the polygon $\Gamma(f)$, then there exists an irreducible bivariate polynomial $g(X,Y)$, such that $A = \Gamma(g)$. Moreover the set of coefficients of f and of g are the same (including their multiplicity) and the curves defined by f and g are birationally equivalent.*

Proof. Let $M = \begin{pmatrix} k & l \\ m & n \end{pmatrix}$ be the matrix such that $M\Gamma(f)$ is a shift of A. Define $g(U,V) = U^\lambda V^\mu \sum_{(a,b)\in S} \alpha_{(a,b)} U^{ak+bl} V^{am+bn}$. Here λ and μ are so that $\Gamma(g)$ is in default position. By construction g has the same nonzero coefficients as f. The birational equivalence between the curves given by g and f is defined by:

$$\phi : Z(g) \longrightarrow Z(f),$$
$$(U,V) \longrightarrow (U^k V^m, U^l V^n).$$

□

Corollary 3.3.5. *Any elliptic curve over $k(t)$, corresponding to a Delsarte surface, is birationally equivalent to an elliptic curve with Newton polygon as in figure 3.2 and corresponding to a Delsarte surface.*

Proof. An arbitrary elliptic curve, that correspond to a Delsarte surface, has a Newton polygon with exactly one interior point. By theorem 3.3.3, this Newton polygon is linearly equivalent to one of the polygons in figure 3.2. By the proposition, there is a curve, corresponding to a Delsarte surface, with this Newton polygon, that is birationally equivalent to the curve we started with. □

We intend to compute the maximal rank of elliptic Delsarte surfaces. Since the rank is invariant under birational maps the corollary says that we only need to compute the maximal rank of each elliptic Delsarte surface, with a Newton polygon as in figure 3.3.3.

3.4 All Delsarte surfaces.

For each of the Newton polygons we can easily determine what kind of elliptic Delsarte surfaces have a corresponding elliptic curve with this polygon as its Newton polygon. For each Newton polygon we want to compute the maximal rank of these curves. To do this we only have to consider some of the elliptic Delsarte surfaces corresponding to this Newton polygon. This is best illustrated by an example.

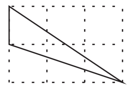


Figure 3.3: A polygon with one interior point.

Example 3.4.1. Consider E an elliptic Delsarte surface with corresponding Newton polygon as in Figure 3.4. Then it is defined by one of the following equations:

$$f_1 = (t^a + t^d)Y + t^b X^3 + t^c Y^2,$$

$$f_2 = t^a Y + (t^b + t^d)X^3 + t^c Y^2,$$
$$f_3 = t^a Y + t^b X^3 + (t^c + t^d)Y^2,$$
$$f_4 = t^a Y + t^b X^3 + t^c Y^2 + t^d XY.$$

Here a, b, c, d are non-negative integers.

If we extend the field over which these curves are defined from $k(t)$ to $k(s)$, with $s^3 = t$, the rank can only increase. So to find the maximal rank we can assume that the constants a, b, c, d are divisible by 3.

By the transformation $\xi = t^{(b+c-2a)/3}X$ and $\eta = t^{c-a}Y$ we see that each of these curves is isomorphic to one of the curve defined by the following equations

$$\tilde{f}_1 = (1 + t^n)Y + X^3 + Y^2,$$

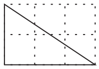
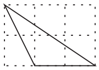
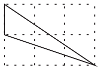
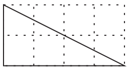
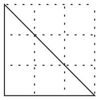
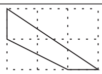
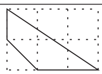
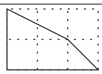
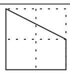
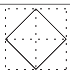
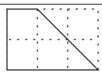
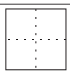
$$\tilde{f}_2 = Y + (1 + t^n)X^3 + Y^2,$$
$$\tilde{f}_3 = Y + X^3 + (1 + t^n)Y^2,$$
$$\tilde{f}_4 = Y + X^3 + Y^2 + t^n XY.$$

We will denote the curve defined by $\tilde{f}_1$, $\tilde{f}_2$, $\tilde{f}_3$ and $\tilde{f}_4$ by E_n^{3a}, E_n^{3b}, E_n^{3c} and E_n^{3d} respectively. In the next chapter we will compute the maximal rank of these families of elliptic curves.

We do exactly the same as in example 3.4.1 for all the polygons in figure 3.2. The polygons with five or six corners do not occur as Newton polygons corresponding to Delsarte surfaces. All the other polygons have been placed into the following table. For each polygon we described some series of curves. The maximal rank for each of the polygons is attained in one of these series.

Note that there is a choice in the construction of this table. In the cases where the polynomial has four terms over the field $k(t)$ the term t^n can without loss of generality be placed in front of any of the terms in the polynomial. The choices made here are partially arbitrary and have partially been motivated by ease of computing the rank.

The computations done to fill the remainder of the table can be found in the rest of this chapter and the next chapter.

Picture	Name	Form	Maximal rank	Occurring for n
	E_n^{1a}	$1+t^n+X^3+Y^2$	68	360
	E_n^{1b}	$1+t^nX+X^3+Y^2$	56	840
	E_n^{1c}	$1+t^nX^2+X^3+Y^2$	9	20
	E_n^{1d}	$1+(1+t^n)X^3+Y^2$	18	60
	E_n^{1e}	$t^n+Y+X^3+Y^2$	68	360
	E_n^{1f}	$1+t^nXY+X^3+Y^2$	9	10
	E_n^{1g}	$1+X^3+(1+t^n)Y^2$	4	6
	E_n^{2a}	$(1+t^n)X+X^3+Y^2$	24	24
	E_n^{2b}	$t^nX+X^2+X^3+Y^2$	3	12
	E_n^{2c}	$X+(1+t^n)X^3+Y^2$	24	24
	E_n^{2d}	$t^nX+XY+X^3+Y^2$	3	12
	E_n^{2e}	$X+X^3+(1+t^n)Y^2$	6	12
	E_n^{3a}	$(1+t^n)Y+X^3+Y^2$	18	60
	E_n^{3b}	$Y+(1+t^n)X^3+Y^2$	18	60
	E_n^{3c}	$Y+X^3+(1+t^n)Y^2$	18	60
	E_n^{3d}	$Y+t^nXY+X^3+Y^2$	1	2
	E_n^{4a}	$1+t^n+X^4+Y^2$	24	24
	E_n^{4b}	$1+t^nX+X^4+Y^2$	56	840
	E_n^{4c}	$t^n+X^2+X^4+Y^2$	3	12
	E_n^{4d}	$1+t^nX^3+X^4+Y^2$	56	840
	E_n^{4e}	$1+(t^n+1)X^4+Y^2$	24	24
	E_n^{4f}	$1+t^nY+X^4+Y^2$	24	12
	E_n^{4g}	$t^n+XY+X^4+Y^2$	3	12
	E_n^{4h}	$1+t^nX^2Y+X^4+Y^2$	24	12
	E_n^{4i}	$1+X^4+(1+t^n)Y^2$	6	12
	E_n^{5a}	$1+t^n+X^3+Y^3$	18	60
	E_n^{5b}	$1+t^nX+X^3+Y^3$	68	120
	E_n^{5c}	$1+t^nX^2+X^3+Y^3$	68	120
	E_n^{5d}	$1+(1+t^n)X^3+Y^3$	18	60
	E_n^{5e}	$1+t^nY+X^3+Y^3$	68	120
	E_n^{5f}	$1+t^nXY+X^3+Y^3$	1	2
	E_n^{5g}	$1+t^nX^2Y+X^3+Y^3$	68	120
	E_n^{5h}	$1+t^nY^2+X^3+Y^3$	68	120
	E_n^{5i}	$1+t^nXY^2+X^3+Y^3$	68	120
	E_n^{5j}	$1+X^3+(1+t^n)Y^3$	18	60
	E_n^{6}	$t^nX^2+Y+X^3+Y^2$	9	20
	E_n^{7}	$t^nX+Y+X^3+Y^2$	56	840
	E_n^{8}	$1+t^nX^2Y+X^3+Y^2$	56	560
	E_n^{9}	$t^n+X^2Y+X^2+Y^2$	3	12
	E_n^{10}	$t^nX+Y+X^2Y+XY^2$	0	1
	E_n^{11}	$t^n+XY^2+X^3+Y^2$	18	120
	E_n^{12}	$t^n+X^2+Y^2+X^2Y^2$	0	1

3.5 Equivalence

The above table describes 42 families of elliptic curves. In the first chapter we saw that two k-isogenous elliptic curves have the same rank. To reduce the amount of work in computing the maximal rank of these families, it is useful to identify as many k-isogenies as possible. This is what we will do in this section.

3.5.1 Families related to E_n^{1a}

Here we explain the relation between the families of curves E_n^{1a}, E_n^{1e}, E_n^{5b}, E_n^{5c}, E_n^{5e}, E_n^{5g}, E_n^{5h} and E_n^{5i}.

Permuting homogeneous coordinates X, Y, Z gives isomorphisms between the families described by E_n^{5b}, E_n^{5c}, E_n^{5e}, E_n^{5g}, E_n^{5h} and E_n^{5i}.

A short Weierstrass form for the curves E_n^{1e} is

$$1 - 4t^n + \xi^3 + \eta^2 = 0.$$

The field automorphism defined by $t \to \sqrt[n]{-4}t$ yields the form described by E_n^{1a}. We conclude that the curves E_n^{1a} and E_n^{1e} are k-isomorphic.

Using the isomorphism given by $V = \sqrt{3}\frac{X^2-Y^2}{4} + \frac{t^n}{4\sqrt{3}}$ and $U = \frac{X+Y}{2}$ we see that the curve E_n^{5b} is isomorphic to the curve given by:

$$V^2 + U^4 + \frac{1}{2}t^n U^2 + \frac{1}{2}U - \frac{1}{48}t^{2n} = 0.$$

Using ideas described in [4, Ch. 8] one finds that this curve is isomorphic to the curve given by:

$$\eta^2 + \xi^3 + 1 + \frac{4}{27}t^{3n} = 0.$$

The field automorphism defined by $t \to \sqrt[3n]{4/27}t$ brings this precisely to the curve E_{3n}^{1a}. From this it follows that E_{3n}^{1a} and E_n^{5b} are k-isomorphic.

We conclude that the curves E_{3n}^{1a}, E_{3n}^{1e}, E_n^{5b}, E_n^{5c}, E_n^{5e}, E_n^{5g}, E_n^{5h} and E_n^{5i} are all k-isomorphic, and hence have the same rank.

3.5.2 Families related to E_n^{1b}

Here we show that E_{3n}^{1b}, E_{3n}^{4b}, E_{3n}^{4d}, E_{3n}^{7} and E_{2n}^{8} are k-isomorphic. From this we can conclude that these families of curves have the same maximal rank.

The short Weierstrass form of E_n^7 is

$$1 + \frac{4}{\sqrt[3]{-4}}t^n\xi + \xi^3 + \eta^2 = 0.$$

The field automorphism defined by $t \to (-4)^{2/(3n)}t$ brings this in the form described by E_n^{1b}.

Using ideas from [4, Ch. 8] we find that the curve E_n^{4b} is isomorphic to the one given by:

$$\eta^2 + \frac{t^n(i-1)}{2}\eta + \xi + \xi^3 = 0.$$

After an automorphism of $k(t)$ this gives exactly the curve E_n^7.

The curve of the form E_n^{4b} and E_n^{4d} are isomorphic. An the isomorphism is given by $(X,Y) \mapsto (\frac{1}{X}, \frac{Y}{X^2})$.

The curve E_{2n}^8 is isomorphic to the curve given by:

$$\eta^2 + 1 + \xi^3 - \frac{t^{4n}}{4}\xi^4 = 0.$$

There is an automorphism of $k(t)$ bringing this precisely to E_{3n}^{4d}.

3.5.3 Families related to E_n^{1c}

We will show that the curves E_{2n}^{1c}, E_n^{1f} and E_{2n}^6 are k-isomorphic.

The curve E_n^{1f} is isomorphic to the curve given by:

$$\eta^2 + 1 + \xi^3 - \frac{1}{4}t^{2n}\xi^2 = 0.$$

There is an automorphism of $k(t)$ bringing this curve precisely to E_{2n}^{1c}. Likewise there is an isomorphism from E_n^6 to the curve given by

$$\eta^2 + 1 + \xi^3 + \sqrt[3]{-4}t^n\xi^2 = 0,$$

and there is an automorphism of $k(t)$ sending this curve to E_n^{1c}.

3.5.4 Families related to E_n^{1d}

We will show that the curves E_n^{1d}, E_n^{3a}, E_n^{3b}, E_n^{3c}, E_n^{5a}, E_n^{5d} and E_n^{5j} are isomorphic.

Permuting homogeneous coordinates X, Y, Z gives isomorphisms between the curves E_n^{5a}, E_n^{5d} and E_n^{5j}.

The curves E_n^{3a}, E_n^{3b} and E_n^{3c} are isomorphic with morphisms $(X,Y) \mapsto (X,(1+t^n)Y)$ from E_n^{3c} to E_n^{3b}, and $(X,Y) \mapsto ((1+t^n)X,(1+t^n)Y)$ from E_n^{3b} to E_n^{3a}.

Moreover, there is an isomorphism from E_n^{3b} to E_n^{1d} given by $(X,Y) \mapsto (\sqrt[3]{-4}X, \sqrt{-1}(2Y+1))$.

Using ideas from [4, Ch. 8] we find that the curve E_n^{5a} is isomorphic to E_n^{1d}.

3.5.5 Families related to E_n^{2a}

We will show that the curves E_{2n}^{2a}, E_{2n}^{2c}, E_{2n}^{4a}, E_{2n}^{4e}, E_n^{4f} and E_n^{4h} are k-isomorphic.

The curves E_n^{2a} and E_n^{2c} are isomorphic, with an isomorphism from E_n^{2a} to E_n^{2c} given by $(X,Y) \mapsto (\frac{1}{X}, \frac{Y}{X^2})$. The curves E_n^{4a} and E_n^{4e} are also isomorphic, with isomorphism given by $(X, Y \mapsto)(\frac{1}{X}, \frac{Y}{X^2})$. The curves E_n^{4f} and E_n^{4h} are likewise isomorphic, with isomorphism given by $(X,Y) \mapsto (\frac{1}{X}, \frac{Y}{X^2})$.

Using [4, Ch. 8] we bring E_n^{4a} in short Weierstrass. This gives after some calculation the curve E_n^{2a}. So these two forms are also isomorphic. Alternatively, $(X,Y) \mapsto (X^2, XY)$ defines an isogeny from E_n^{4a} to E_n^{2a}.

Using the map $(X,Y) \mapsto (X, Y + \frac{1}{2}t^n)$ we see that E_n^{4f} is isomorphic to the curve given by:

$$\xi^4 + \eta^2 + 1 - \frac{1}{4}t^{2n} = 0.$$

There is an automorphism of $k(t)$ bringing this precisely in the form E_{2n}^{4a}.

3.5.6 Families related to E_n^{2b}

We show here that the curves E_n^{2b}, E_n^{2d}, E_n^{4c}, E_n^{4g} and E_n^{9} are k-isogenous and hence have the same rank.

The curve E_n^{2d} is isomorphic to the curve defined by:

$$\eta^2 + \xi^3 + \xi^2 + 16t^n\xi = 0.$$

There is an automorphism of $k(t)$ sending this curve to E_n^{2b}, so E_n^{2d} and E_n^{2b} are k-isomorphic.

There is an isogeny from E_n^{4g} to E_n^{2d} given by: $(X, Y) \mapsto (X^2, XY)$, so these two curves are isogenous.

There is an isomorphism from E_n^{4g} to the curve given by

$$\eta^2 + \xi^4 + \xi^2 + 16t^n = 0.$$

There is an automorphism of $k(t)$ sending this curve to E_n^{4c}.

Likewise there is is an isomorphism from E_n^{9} to the curve given by

$$\eta^2 - \frac{1}{4}t^n + \xi^2 + \xi^4 = 0.$$

There is an automorphism of $k(t)$ sending this curve to E_n^{4c}.

3.5.7 Families related to E_n^{2e}

The curves E_n^{2e} and E_n^{4i} are isomorphic. An isomorphism from E_n^{4i} to E_n^{2e} is given by: $(X, Y) \mapsto (\frac{\sqrt[4]{-1}-X}{\sqrt[4]{-1}+X}, \frac{\sqrt{-2}Y}{(X+\sqrt[4]{-1})^2})$.

3.5.8 Families related to E_n^{3d}

There is an isogeny from E_n^{5f} to E_n^{3d}, given by $(X, Y) \mapsto (XY, Y^3)$. Hence the curves E_n^{5f} and E_n^{3d} are isogenous.

3.5.9 Families related to E_n^{12}

There is also an isogeny from E_n^{12} to E_n^{10}, given by $(X, Y) \mapsto (XY^{-1}, XY)$, hence the curves E_n^{10} and E_n^{12} are isogenous.

3.5.10 Conclusion

We see that up to k-isogeny there are only 11 families of elliptic Delsarte surfaces. In each of these families we will identify an elliptic Delsarte surface of which the rank is maximal within that family. This will give 11 elliptic Delsarte surfaces. These are placed in the following table with their Mordell-Weil rank and some information on their j-invariant.

Curve	Rank	j-invariant
E_{360}^{1a}	68	0
E_{840}^{1b}	56	non-constant
E_{20}^{1c}	9	non-constant
E_{60}^{1d}	18	0
E_{6}^{1g}	4	0
E_{24}^{2a}	24	1728
E_{12}^{2b}	3	non-constant
E_{12}^{2e}	6	1728
E_{2}^{3d}	1	non-constant
E_{120}^{11}	18	non-constant
E_{1}^{12}	0	non-constant